radius a is then simply $Q = \epsilon_0 E_r [K_r \pi a^2]$ and coupled power $P = (1/2) Z_0 (\omega Q)^2$, where E_r is the peak normal electric field and $Z_0 = 50\ \Omega$. K_r was calculated using a Laplace solver on a CDC-7600 to be 3.846 and was measured to be 3.77.

The coupling was also calculated and measured as the probe was withdrawn into the waveguide wall to obtain as much as 9-dB less coupling than the flush coupling value.

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Computation of Inductance of Simple Vias Between Two Striplines Above a Ground Plane

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Abstract —**In this paper, an analysis is developed for calculating the lumped inductance of a simple via connecting two infinitely thin striplines, located above a perfectly conducting ground plane. The striplines are oriented in the same direction, and the via is assumed to be in the form of an infinitely thin vertical plate, connecting the two lines. This system is analyzed by a hybrid partial-element and circuit-theory approach. Numerical results are presented to illustrate the application of this technique.**

I. INTRODUCTION

In order to provide an accurate analysis of electronic circuits containing transmission lines, it is necessary to take into account the discontinuities of the transmission lines, such as the line terminations, bends, crossovers, and connections between different transmission lines (vias). In the simplest model, these discontinuities are represented by equivalent lumped-element networks, consisting of inductors and capacitors.

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The lumped elements representing the discontinuities are obtained by static analyses (magnetostatic and electrostatic). These elements, essentially, take into account the excess magnetic and electric energy stored in the field in the vicinity of the discontinuity, as compared to the energy stored in the field along a uniform transmission line. Therefore, it is possible in some cases that the inductances or capacitances have negative values.

In this paper, an analysis is developed for calculating the lumped inductance of a simple via connecting two infinitely thin striplines, located above a perfectly conducting ground plane. The striplines are oriented in the same direction, and the via is assumed to be in the form of an infinitesimally thin vertical plate, connecting the two transmission lines.

Only a few papers exist dealing with the analysis of similar structures. A brief survey of the existing techniques is given in [1]. The so-called partial-element method [2]–[3] is based on dividing the conductor into a number of rectangular elements, and the self and mutual inductances of these elements are evaluated. Resistances can also be included in such a procedure. These inductances (and possible resistances) are thereafter interconnected so to form a network, which is solved by standard circuit-theory techniques. In the second group of papers [4]–[5], the integral equations and the Galerkin's method are used to solve for the unknown current distribution in the conductors. Although at first glance these two methods seem to be different, the partial-element method is, basically, very closely related to Galerkin's technique.

In this paper, we essentially employ the partial-element method, given in [3]. A brief description of the method is given in Section II. It is worth mentioning that the method gives, as a byproduct, the inductances per unit length of the two striplines, joined by the via. In Section III, some numerical examples are given, showing the dependence of the via equivalent inductance on the via dimensions.

II. BASIC PRINCIPLES

Let us consider a perfectly conducting body which has two well-defined terminals (i.e., has a distinct port). Our objective is to find the inductance of this structure as seen from the port. The method which we are going to apply for solving this problem is based on the partial-element method, described in [3]. It is a hybrid of the electromagnetic-field methods and the electric-circuit methods. Although we are not going to explicitly use the frequency-domain analysis, our solution corresponds essentially to the limiting case of time-harmonic fields when the frequency tends to zero. This situation is sometimes referred to as the magnetostatic analysis.

First note that the currents in our object are located only in a surface layer, because the structure is perfectly conducting. The surface-current density vector $\boldsymbol{J}_s$ at any point on the surface can be represented as the sum of two orthogonal components. For the sake of simplicity, we shall assume that the surface of the conductor is piecewise flat and that the current-density vector can be represented in terms of two local, say, u and v, components. We first approximate the conductor by surface patches (partial elements) carrying currents of constant density over a patch that will represent the body regarding the magnetic field it produces, and treat these patches as simple inductances. Next, we create a network of these inductances and thus find the total inductance between the input ports. Enforcing the first Kirchhoff's law for all the nodes of the network ensures that the

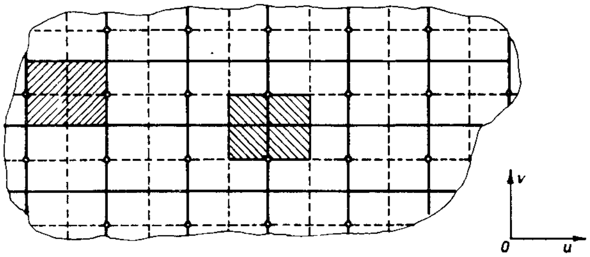

Fig. 1. A portion of the surface of the analyzed body with the partial elements. —— boundaries of partial elements carrying u-directed current --- boundaries of partial elements carrying v-directed current. ///// example of an element carrying u-directed current. \\\\\ example of an element carrying v-directed current.

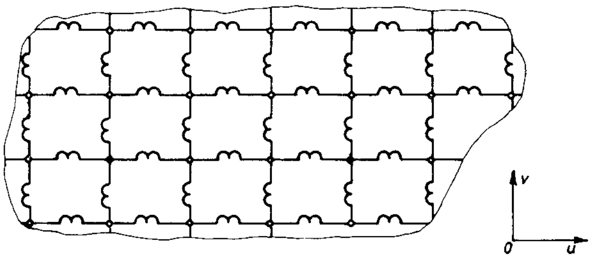

Fig. 2. Inductance network representing the surface of Fig. 1.

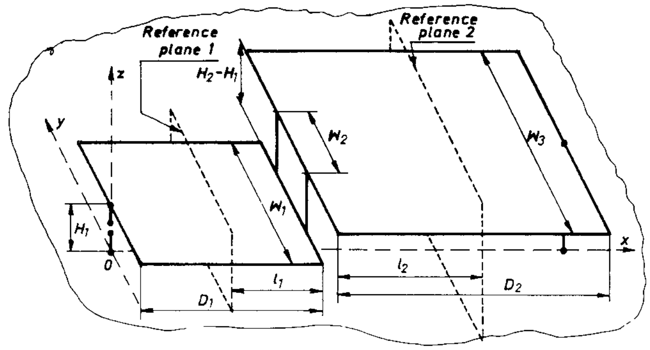

Fig. 3 Sketch of a stripline step above a ground plane. The strips and the via are rectangular, infinitely thin plates. The right end of the second strip is short-circuited to the ground plane, and the left end of the first strip represents the input port.

continuity equation is satisfied, which is always postulated in the magnetostatic analysis. (In other words, we are assuming that no excess charges are localized at the conductor surface.)

Let us specify a set of nodes over the surface, forming a rectangular grid (see Fig. 1). We should take two different schemes for dividing the conductor into partial elements—one for the u-component, and another for the v-component of the surface current, as also sketched in Fig. 1. After replacing the partial elements by the equivalent inductances, we obtain a network as shown in Fig. 2. The inductances parallel to one axis are mutually coupled, while the coupling between mutually perpendicular inductances (i.e., between patches corresponding to mutually perpendicular surface-current components) is zero. Of course, if the whole body is observed, one should take into account all the possible couplings between the patches. It should be noted that the grid need not be rectangular, nor the patches need be rectangles, but the choice taken here eases the computations of the self and mutual inductances, since most of the integrals involved in the analysis can be evaluated explicitly.

The self and mutual inductances can be obtained starting from the general expression for the magnetic energy stored in the magnetic field produced by surface electric currents, of density $\boldsymbol{J}_s$, to yield

$$W_m = \sum_{i=1}^{n} \sum_{i'=1}^{n} \frac{1}{2} \frac{\mu_0}{4\pi} \int_{s_i} \int_{s_{i'}} \frac{\boldsymbol{J}_s(\boldsymbol{r}') \cdot \boldsymbol{J}_s(\boldsymbol{r}) \, ds' \, ds}{|\boldsymbol{r} - \boldsymbol{r}'|} \tag{1}$$

where $s_i, i = 1, \cdots, n$ are the patches into which the body surface is divided.

In the present approach, we are assuming that the current density is uniform over a patch, i.e., that the vector $\boldsymbol{J}_s$ is constant over the patch. The total current over a patch is, hence, wJ_s, where w is the width of the patch, i.e., the dimension of the patch that is perpendicular to the vector $\boldsymbol{J}_s$ of that patch. Thus, the mutual inductance of two arbitrary patches is given by

$$L_{ii'} = \frac{2W_{mii'}}{I_i I_{i'}} = \frac{1}{I_i I_{i'}} \frac{\mu_0}{4\pi} \int_{s_i} \int_{s_{i'}} \frac{\boldsymbol{J}_s(\boldsymbol{r}') \cdot \boldsymbol{J}_s(\boldsymbol{r}) \, ds' \, ds}{|\boldsymbol{r} - \boldsymbol{r}'|}. \tag{2}$$

This equation is valid for any i and i', and thus gives both the mutual and self inductances. The dual surface integral given in (2), which is actually a four-fold integral in terms of line coordinates, can be in principle evaluated either numerically or explicitly, thus yielding the elements of the inductance network shown in Fig. 2. This network can be, thereafter, solved by using the circuit-theory methods to produce the total inductance between the input terminals.

In this paper, the above method is applied to the analysis of a stripline step, which can be considered as a simple via between two striplines, located above a perfectly conducting ground plane (Fig. 3). The orientation of the two striplines is identical, i.e., both of them extend along the x-axis. The via is assumed in the form of a vertical rectangular plate, joining the striplines. The widths of the two lines and the width of the via can be arbitrary, but the system is assumed to possess a plane of symmetry ($0xz$ in Fig. 3). We shall sometimes consider the two strips and the via as three rectangular plates.

For the present method, we must ensure that the overall dimensions of the striplines are finite, and thus we must take finite-length sections of both striplines (D_1, D_2). These lengths should be sufficiently large to ensure that there is a portion in the middle of any of the lines where the current distribution is practically identical to that of an infinitely long line. In another words, the lengths of the lines must be longer than the zones in which the effect of the terminal connections and of the via on the current distribution is significant. In the present analysis, we are not going to take into account the actual shape of the terminals or the short-circuiting conductor at the other end. This approach presents no conceptual difficulties with the circuit-theory methods, but from the field-theory standpoint it is meaningless to take the terminal or the short-circuiting wires to be infinitely thin. However, we are not interested in the properties of the wire junctions, but rather only in the currents and fields away from them, towards the via.

Although, in principle, we can represent the ground plane (over a finite area) by a number of rectangular patches, it is much simpler (and computationally faster) to replace the ground plane by the mirror-image of the original system.

For the structure in hand, we can define rectangular grids of nodes, which are uniform over each of the three segments of the structure shown in Fig. 3, i.e., over the two strips and the via plate. The same grid should be taken for the image in the ground plane.

Regarding computation of the dual surface integrals, given in (2), we can distinguish between two different cases. The first is

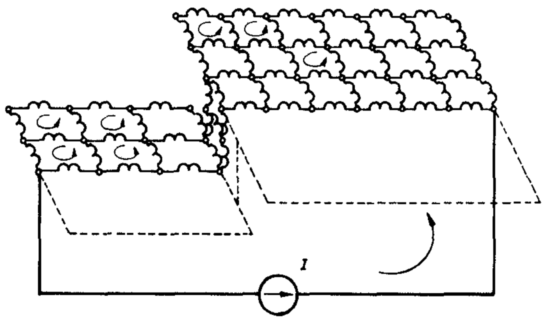

Fig. 4. The equivalent network for the system of Fig. 3. The self and mutual inductances include the influence of the symmetrical other half of the system and of the images.

when the two patches are mutually parallel (and carry currents in the same direction), such as any two patches lying in the two strips, or any two patches lying in the via plate. These integrals can be evaluated explicitly, as shown in the Appendix. The second case is when one of the two patches lies in a horizontal plate (i.e., one of the strips), while the other patch lies in the via plate. These integrals can be evaluated by combining the explicit and numerical integration, as also shown in the Appendix.

Once the self and mutual inductances are obtained, we have to procede to the circuit-theory analysis of the equivalent inductance network. Taking the symmetry into account, we have to consider only the network shown in Fig. 4. We can excite the network by an ideal current generator, of a known amplitude, and apply the mesh equations, choosing the elemental windows for the meshes and orienting them in an anti-clockwise direction. The mesh equations should be set in the frequency domain. However, the term jw can be canceled out in all the equations, so that no information about the frequency is required. This means that the frequency can be taken arbitrarily low, yielding the zero frequency in the limiting case, and thus justifying the term "magnetostatic analysis." Once the mesh equations are solved, the mesh currents are known, and the voltage between any two nodes of the network can easily be found.

In the simplest model, which we are using here, the influence of the via is replaced by an equivalent inductance (regarding the magnetic energy), as shown in Fig. 5. Namely, we assume that both transmission lines (i.e., the two striplines) extend to the very position of the via plate and, in addition, that the inductances per unit length of the two lines are uniform up to the very junction between them. The inductance inserted in this equivalent circuit between the two lines should make up for the differences between this idealization and the reality.

Close to the terminals of the system shown in Fig. 3, the current distribution along the strips deviates significantly from the distribution which would exist on an infinitely long strip. In the vicinity of the via, the current distribution also deviates from the infinite-line case. However, we are interested exactly in the effects of the latter deviation, while the terminals are of no importance for our analysis.

Note that in the two-dimensional case (i.e., for an infinitely long stripline), the voltage between any two points lying in a cross section of the strip is zero. In contrast to this, in the fringing zone, this voltage generally differs from zero. If the two strips in the system of Fig. 3 are of sufficient lengths, then away from the terminal and via zones, around the middles of the strips, the current and potential distributions are practically identical to the corresponding two-dimensional case. If we choose two reference planes, one at each strip, lying in these particular regions,

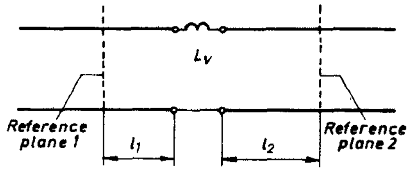

Fig. 5. Equivalent representation of the via shown in Fig. 3.

then we can uniquely define the voltage between these planes. From this voltage, we can obtain the apparent inductance between the two cross sections. Let us denote this inductance by L_a. (This inductance contains no information about the strip terminals, which are left out.)

Referring to Figs. 3 and 5, we can see that the equivalent circuit shown in Fig. 5 should equal the total inductance in the system of Fig. 3 between the cross sections 1 and 2. The total inductance in the equivalent circuit consists of the inductances of the two transmission lines, and the via equivalent inductance L_v. If the inductances per unit length of the two lines are L_1' and L_2', respectively, and if the distances between the reference planes and the via are l_1 and l_2, respectively, we can write

$$L_a = l_1 L_1' + l_2 L_2' + L_v. \tag{3}$$

From (3), we can evaluate L_v if we know L_1' and L_2'. Although these inductances can be obtained from a separate analysis, the results which we have obtained by solving the mesh equations provide sufficient information to calculate L_1' and L_2'. Namely, we can observe a short section of a strip in the middle zone, around the reference plane, and compute the voltage along this section. Of course, since we are in the region where the current and potential distributions are the same as in the two-dimensional case, we can uniquely define this voltage. From this voltage, we get the apparent inductance of this short section between the two cross sections, and hence the inductance per unit length of the transmission line. Finally, from (3), we can compute the inductance L_v, which is the final goal of the present analysis.

At this point, it should be mentioned that the number of partial elements should be taken sufficiently large so that the variation of the current distribution over all the three plates can be represented with a high degree of accuracy. However, there is no *a priori* rule about this choice and it is always advisable to solve the same problem for various numbers of nodes and compare the results. Regarding the choice of the reference planes, maybe the simplest choice is to take them as close as possible to the very middle of the strips. Some insight into the numerical stability of the present method can be gained from the next section.

III. Numerical Examples

In all the examples, the lengths of the two strips are taken to be $D_1 = D_2 = 20$ mm, but the other dimensions of the system, as well as the numbers of partial elements, are varied. The results are presented in Table I. In this table, the symbols have the following meanings:

- H_1 distance between the perfectly conducting plane and the first strip,
- H_2 distance between the perfectly conducting plane and the second strip,
- W_1 width of the first strip,
- W_2 width of the via plate,
- W_3 width of the second strip,
- NX_1 number of meshes along the length of the first strip,
- NX_2 number of meshes along the height of the via plate,

TABLE I
EQUIVALENT VIA INDUCTANCE FOR VARIOUS DIMENSIONS AND NUMBERS OF PARTIAL ELEMENTS FOR THE SYSTEM SKETCHED IN FIG. 3

H_1	H_2	W_1	W_2	W_3	NX_1	NX_2	NX_3	NY_1	NY_2	NY_3	L'_1	L'_2	L_v
0.01	1	5	5	5	4	2	4	4	4	4	2.50	165 1	85 1
0.01	1	5	5	5	6	2	6	4	4	4	2.50	164.5	90 4
0 01	1	5	5	5	16	4	16	4	4	4	2.50	164.4	91.3
1	2	5	5	5	6	2	6	4	4	4	165.7	256.7	107.1
1	2	5	5	5	16	4	16	4	4	4	165 9	257.1	101.5
0.5	1	5	2.5	10	8	2	8	2	1	4	98 5	98.2	129.9
0 5	1	5	2.5	10	8	2	8	4	2	8	97.7	97.6	140.5
0 5	1	5	2.5	10	12	4	12	4	2	8	97 7	97.2	158.3
0 5	1	5	1.25	10	12	4	12	4	1	8	97.7	97.2	297.9
0.5	1	5	2.5	10	12	4	12	4	2	8	97.7	97.2	158.3
0.5	1	5	5	10	12	4	12	4	4	8	97.7	97.3	79.0

NX_3 number of meshes along the length of the second strip,
NY_1 number of meshes along the half-width of the first strip,
NY_2 number of meshes along the half-width of the via plate,
NY_3 number of meshes along the half-width of the second strip,
L'_1 inductance per unit length of the first strip (nH/m),
L'_2 inductance per unit length of the second strip (nH/m),
L_v equivalent via inductance (pH).

The entries in Table I are grouped so that they can easily be compared. All the dimensions of the system are in millimeters.

For the first three entries, the first strip is extremely close to the ground plane. The entries differ only in the number of partial elements used in the computation. Even with the smallest number, the results for the inductances are very good. The second group of entries is similar, except that the two strips are placed at higher levels. The conclusions regarding the quality of the results are similar as before. Note that the second strip in the first group is identical to the first strip in the second group, which results in very close corresponding results for the inductances per unit length in the two cases.

In the third group of entries, the widths of the two strips, as well as that of the via plate, are unequal. Again, the number of partial elements is varied, and this introduces only insignificant changes into the inductances per unit length. The via equivalent inductance, however, suffers somewhat greater changes, which could be expected, because of the fringing effects at the connection between the via plate and the two strips. Of course, in order to properly model the current distribution in the vicinity of these junctions, we need a large number of partial elements. Note that with the increasing number of the partial elements, the via equivalent inductance increases because of progressively better modeling of the fringing effects.

Finally, in the fourth group of entries, the via width is progressively increased, while the other dimensions are kept intact. For convenience, the last entry from the third group is repeated as the second entry in the fourth group. As expected, the via equivalent inductance increases as the via width decreases and the inductances per unit length of the two transmission lines are practically constant.

V. CONCLUSION

This paper presents a method which can be used for magnetostatic analysis of perfectly conducting three-dimensional structures. The method is applied to analysis of a flat via, connecting two zero-thickness striplines, placed above a ground plane. Essentially, the surface of the conductor is replaced by rectangular partial elements. These elements are characterized by the self and mutual inductances. These equivalent inductances are interconnected in a network, which is solved by the circuit-theory methods. From the known currents through the inductances, the inductances per unit length of the two striplines are computed, and the equivalent via inductance is obtained as the final result. Numerical results are presented showing a good stability and reasonably accurate values of the via inductance even if only a relatively small number of partial elements is employed.

APPENDIX
EVALUATION OF INTEGRALS

As already mentioned in Section II, in the present analysis there exist two kinds of integrals which should be evaluated in (2) in order to find the self and mutual inductances of the rectangular partial elements.

The first kind of integrals appear when both partial elements are parallel to a coordinate plane. This is the case if the two elements belong to the strips or their images, or if the two elements belong to the via, or its image. Let us suppose that both rectangular elements are parallel to the $0xy$ plane, and that their edges are parallel to the x and y axes. Let the location of the first element be determined by its vertices (x_{l1}, y_{l1}, z_1), (x_{u1}, y_{l1}, z_1), (x_{u1}, y_{u1}, z_1), (x_{l1}, y_{u1}, z_1), while the vertices of the second element are (x_{l2}, y_{l2}, z_2), (x_{u2}, y_{l2}, z), (x_{u2}, y_{u2}, z_2), (x_{l2}, y_{u2}, z_2). Noting that the surface-current densities in (2) are constant vectors over the elements, it is easy to see that the integral that has to be evaluated has the form

$$G = \int_{x_{l1}}^{x_{u1}} \int_{x_{l2}}^{x_{u2}} \int_{y_{l1}}^{y_{u1}} \cdot \int_{y_{l2}}^{y_{u2}} \frac{1}{\left[(x-x')^2+(y-y')^2+(z_1-z_2)^2\right]^{1/2}} \, dy' \, dy \, dx' \, dx. \tag{4}$$

This four-fold integral can be reduced to 16 two-fold integrals by introducing the following changes of variables: $s = x' - x, s' = x', t = y' - y, t' = y'$, i.e.,

$$G = \sum_{p=1}^{4} \sum_{q=1}^{4} a_p b_q H(u_p, v_q) \tag{5}$$

where

$$H(u,v) = \int_0^u \int_0^v \frac{(u-s)(v-t)}{(s^2+t^2+z^2)^{1/2}} \, dt \, ds \tag{6}$$

and $u_1 = |x_{u2} - x_{l1}|$, $u_2 = |x_{l2} - x_{u1}|$, $u_3 = |x_{u2} - x_{ul}|$, $u_4 = |x_{l2} - x_{l1}|$, $v_1 = |y_{u2} - y_{l1}|$, $v_2 = |y_{l2} - y_{u1}|$, $v_3 = |y_{u2} - y_{u1}|$, $v_4 = |y_{l2} - y_{l1}|$, $a_1 = a_2 = b_1 = b_2 = 1$, $a_3 = a_4 = b_3 = b_4 = -1$, and $z = z_2 - z_1$.

The integral $H(u,v)$ can be evaluated explicitly, to yield

$$H(u,v) = uv\left\{ z\left[\arcsin\frac{uz}{(u^2+v^2)^{1/2}(v^2+z^2)^{1/2}} + \arcsin\frac{vz}{(u^2+v^2)^{1/2}(u^2+z^2)^{1/2}} - \frac{\pi}{2}\right] - \frac{u}{2}\log\frac{(u^2+v^2+z^2)^{1/2}-v}{(u^2+v^2+z^2)^{1/2}+v} - \frac{v}{2}\log\frac{(u^2+v^2+z^2)^{1/2}-u}{(u^2+v^2+z^2)^{1/2}+u}\right\}$$
$$- \frac{u}{2}\left[u(u^2+v^2)^{1/2} + (v^2+z^2)\log\frac{u+(u^2+v^2)^{1/2}}{(v^2+z^2)^{1/2}} - u(u^2+z^2)^{1/2} - z^2\log\frac{u+(u^2+z^2)^{1/2}}{z}\right]$$
$$- \frac{v}{2}\left[v(u^2+v^2)^{1/2} + (u^2+z^2)\log\frac{v+(u^2+v^2)^{1/2}}{(u^2+z^2)^{1/2}} - v(v^2+z^2)^{1/2} - z^2\log\frac{v+(v^2+z^2)^{1/2}}{z}\right] + \frac{1}{3}\left[(u^2+v^2)^{3/2} - (u^2+z^2)^{3/2} - (v^2+z^2)^{3/2} + z^3\right]. \quad (7)$$

The second kind of integrals appear when one of the partial elements is parallel to the $0xy$ plane, and the second one is parallel to the $0yz$ plane. This is the case when one of the elements belongs to one of the stips, or its image, while the other element belongs to the via, or its image. If the first element vertices are (x_{l1}, y_{l1}, z_1), (x_{u1}, y_{l1}, z_1), (x_{u1}, y_{u1}, z_1), (x_{l1}, y_{u1}, z_1), and the second element vertices are (x_2, y_{l2}, z_{l2}), (x_2, y_{u2}, z_{l2}), (x_2, y_{u2}, z_{u2}), (x_2, y_{l2}, z_{u2}), then the integral to be evaluated has the form

$$G = \int_{x_{l1}}^{x_{u1}}\int_{z_{l2}}^{z_{u2}}\int_{y_{l1}}^{y_{u1}} \cdot \int_{y_{l2}}^{y_{u2}} \frac{1}{\left[(x-x_2)^2+(y-y')^2+(z_1-z)^2\right]^{1/2}}\,dy'\,dy\,dz\,dx. \quad (8)$$

Again, by introducing changes of variables $t = y' - y, t' = y'$, the two integrals over y and y' are reduced to only one integral over t. That integral and the integral over, say x, can be evaluated explicitly, thus yielding

$$G = \int_{z_1}^{z_2} dz \sum_{p=1}^{2}\sum_{q=1}^{4} a_p b_q H(u_p, v_q, z) \quad (9)$$

where

$$H(u,v,z) = \int_0^u\int_0^v \frac{v-t}{(s^2+t^2+z^2)^{1/2}}\,dt\,ds \quad (10)$$

and $a_1 = -1, a_2 = 1, u_1 = |x_{l1} - x_2|, u_2 = |x_{u1} - x_2|$, and b_q and v_q are the same as defined with (6). The integral $H(u,v,z)$ can be integrated explicitly to yield

$$H(u,v,z) = v\left\{ z\left[\arcsin\frac{uz}{(u^2+v^2)^{1/2}(v^2+z^2)^{1/2}} + \arcsin\frac{vz}{(u^2+v^2)^{1/2}(u^2+z^2)^{1/2}} - \frac{\pi}{2}\right] - \frac{u}{2}\log\frac{(u^2+v^2+z^2)^{1/2}-v}{(u^2+v^2+z^2)^{1/2}+v} - \frac{v}{2}\log\frac{(u^2+v^2+z^2)^{1/2}-u}{(u^2+v^2+z^2)^{1/2}+u}\right\}$$
$$- \frac{1}{2}\left[u(u^2+v^2)^{1/2} + (v^2+z^2) \cdot\log\frac{u+(u^2+v^2)^{1/2}}{(v^2+z^2)^{1/2}} - u(u^2+z^2)^{1/2} - z^2\log\frac{u+(u^2+z^2)^{1/2}}{z}\right]. \quad (11)$$

and the remaining integral in (9) can be evaluated numerically.

Note that due to the reciprocity properties, $L_{ii'} = L_{i'i}$, and thus substantial savings in the computation are possible.

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Analyzing Lossy Radial-Line Stubs

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Abstract —**Equations for the design and analysis of lossless radial-line stubs are available in the literature. However, when actually fabricated in microstrip or stripline, these stubs possess finite conductor loss. This attenuation must be included if these components are to be properly integrated with other lossy transmission-line elements as part of a micro-**

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